BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 11

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CONTENTS

I. REVIEW

Last time we:

(1) Described the bijection between morphisms $F : X \to Y$ of Riemann surfaces of degree d branched over $B \subseteq Y$ and monodromy representations $\rho : \pi_1(Y \setminus B, y) \to S_d$.
(2) Studied the particular case where $Y = \mathbb{P}^1$, in which case the monodromy representation ρ can then be described by a collection of permutations. [Review using whiteboard to draw $\mathbb{P}^1$ minus points.]

II. FUNCTION FIELDS

I'm not a fan of the presentation of function fields in Girondo and González-Diez, so today we'll be following chapter 8 of Forster's *Lectures On Riemann Surfaces*.

II.1. **Symmetric functions.** Recall that for a Riemann surface X, $\mathcal{M}(X)$ is the field of meromorphic functions on X. An element $f \in \mathcal{M}(X)$ is a mermorphic function $f : X \to \mathbb{C}$, and we saw that such a function can be viewed as a morphism $X \to \widehat{\mathbb{C}}$.

Given a nonconstant morphism $\pi : Y \to X$ of Riemann surfaces and $f \in \mathcal{M}(X)$, then $f \circ \pi : Y \xrightarrow{\pi} X \xrightarrow{f} \widehat{\mathbb{C}}$ is a morphism, hence can be viewed as a meromorphic function on Y. Thus we get a field morphism

$$\begin{aligned} \pi^* : \mathcal{M}(X) &\to \mathcal{M}(Y) \\ f &\mapsto f \circ \pi \,. \end{aligned}$$

We often consider $\mathcal{M}(X)$ as a subfield of $\mathcal{M}(Y)$ by identifying it with its image $\pi^*(\mathcal{M}(X))$.

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Example 1. Let $Y = X = \widehat{\mathbb{C}}$ and consider the morphism

$$\begin{aligned} \pi : \widehat{\mathbb{C}} &\to \widehat{\mathbb{C}} \\ z &\mapsto z^3 . \end{aligned}$$

Given a meromorphic function $f \in \mathcal{M}(X)$, then $\pi^*(f)(z) = f \circ \pi(z) = f(z^3)$. Thus the corresponding extension of function fields is $\mathbb{C}(z) \supseteq \mathbb{C}(z^3)$.

$$\begin{array}{ccc} Y = \widehat{\mathbb{C}} & \qquad\qquad & \mathbb{C}(z) \\ \downarrow{\scriptstyle \pi} & & \big|{\scriptstyle \pi^*} \\ X = \widehat{\mathbb{C}} & & \mathbb{C}(z^3) \end{array}$$

Remark 2. The associations $X \mapsto \mathcal{M}(X)$, $\pi \mapsto \pi^*$ define a contravariant functor from the category of Riemann surfaces to the category of fields. We will later see that this is in fact an equivalence when we restrict the target category to function fields of one variable.

Let X and Y be Riemann surfaces and $\pi : Y \to X$ be an unramified covering map of degree d, and let $f \in \mathcal{M}(Y)$ be a meromorphic function. Then each point $x \in X$ has an evenly covered open neighborhood U, so $\pi^{-1}(U) = \bigsqcup_{j=1}^{d} V_j$, and the restrictions $\pi|_{V_j} : V_j \to U$ are isomorphisms of Riemann surfaces. Let $\tau_j : U \to V_j$ be the inverse of $\pi|_{V_j}$, and let

$$f_j := \tau_j^* f = f \circ \tau_j \in \mathcal{M}(U) .$$

$$\begin{array}{ccc} V_j & \hookrightarrow & Y \\ {\scriptstyle \tau_j}\uparrow & & \downarrow{\scriptstyle \pi} \\ U & \hookrightarrow & X \end{array} \qquad\qquad \begin{array}{ccc} f|_{V_j} \in \mathcal{M}(V_j) & \hookleftarrow & \mathcal{M}(Y) \ni f \\ \big| & & {\scriptstyle \pi^*}\big| \\ f_j \in \mathcal{M}(U) & \hookleftarrow & \mathcal{M}(X) \end{array}$$

Let T be a variable and define a polynomial in $\mathcal{M}(U)[T]$ by

$$\prod_{j=1}^{d}(T - f_j) = T^n + c_1 T^{n-1} + \cdots + c_d .$$

Then the c_j are meromorphic functions on U and

$$c_j = (-1)^j s_j(f_1, \ldots, f_d)$$

where s_j is the j^{th} elementary symmetric functions in d variables. One can show that these c_j are independent of the choice of neighborhood U of x. Thus we can cover X with evenly covered neighborhoods and glue together the locally defined c_j to obtain meromorphic functions $c_1, \ldots, c_d \in \mathcal{M}(X)$ on all of X.

One can show that these signed symmetric functions can also be defined when $\pi : Y \to X$ is a nonconstant morphism of Riemann surfaces (which may have ramification points). The strategy to prove this is one we've seen before: throw out the ramification values and their preimages to obtain an unramified covering map $\pi|_{Y^*} : Y^* \to X^*$. Since this restriction is unramified, then we know that the signed symmetric functions c_j of f exist in this case. Then one argues that these c_j can be meromorphically continued to the ramification values.

II.2. **Degree of the function field extension induced by a morphism.**

Theorem 3. *Suppose X and Y are Riemann surfaces and $\pi : Y \to X$ is a morphism of degree d. If $f \in \mathcal{M}(Y)$ and $c_1, \ldots, c_d \in \mathcal{M}(X)$ are the signed elementary symmetric functions of f, then*

$$f^d + (\pi^* c_1) f^{d-1} + \cdots + (\pi^* c_{d-1}) f + \pi^* c_d = 0\,. \tag{1}$$

The monomorphism $\pi^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ is an algebraic field extension of degree d.

Proof. We'll just show that $[\mathcal{M}(Y) : \mathcal{M}(X)] \leq d$; to show equality requires the fact that the functions on a Riemann surface separate points. We will show that the lefthand side of (1) is the zero function on Y. Given $y \in Y$, then there exists some $k \in \{1, \ldots, d\}$ such that $\tau_k \circ \pi(y) = y$. Then

$$\begin{aligned}(f^d + (\pi^* c_1) f^{d-1} + \cdots + (\pi^* c_{d-1}) f + \pi^* c_d)(y) &= \prod_{j=1}^{d} (f(y) - \pi^* f_j(y)) \\ &= \prod_{j=1}^{d} (f(y) - f \circ \tau_j \circ \pi(y)) = 0\end{aligned}$$

since one of the factors is

$$f(y) - f \circ \tau_k \circ \pi(y) = f(y) - f(y) = 0\,.$$

Thus every element $f \in \mathcal{M}(Y)$ is the root of a polynomial in $\mathcal{M}(X)[T]$ of degree at most d.

Let $L = \mathcal{M}(Y)$ and $K = \pi^* \mathcal{M}(X)$. Suppose $f_0 \in \mathcal{M}(Y)$ is an element such that the degree d_0 of its minimal polynomial is maximal. We claim that $L = K(f_0)$. Given $g \in L$, consider the extension $K(f_0, g)$. By the Primitive Element Theorem there exists $h \in L$ such that $K(f_0, g) = K(h)$. Since d_0 is maximal, then $[K(h) : K] \leq d_0$. On the other hand,

$$[K(h) : K] = [K(f_0, g) : K] = [K(f_0, g) : K(f_0)][K(f_0) : K] \geq [K(f_0) : K] = d_0\,.$$

Thus

$$K(f_0, g) = K(h) = K(f_0)$$

so $g \in K(f_0)$, hence $L = K(f_0)$. Thus

$$[\mathcal{M}(Y) : \mathcal{M}(X)] = [L : K] = d_0 \leq d\,.$$

□

Definition 4. A covering map $p : Y \to X$ of topological spaces is Galois or normal if for every pair of points $y_0, y_1 \in Y$ with $p(y_0) = p(y_1)$ (i.e., in the same fiber) there exists a deck transformation $\sigma : Y \to Y$ such that $\sigma(y_0) = y_1$.

Remark 5.

- In other words, the covering is Galois if the group of deck transformations acts transitively on every fiber. This is analogous to the case of fields. Let $f \in F[x]$ be an irreducible polynomial, and let $K = F(\alpha)$ where α is a root of f. The extension K/F of fields is Galois iff $\mathrm{Aut}(K/F)$ acts transitively on the roots of f.

- There is an alternative characterization of Galois covering maps using the fundamental group. Choose a basepoint $y_0 \in p^{-1}(x_0)$. Then the covering map $p : Y \to X$ induces a group homomorphism $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ The cover $p : Y \to X$ is Galois iff $p_*(\pi_1(Y, y_0))$ is a normal subgroup of $\pi_1(X, x_0)$.

  This is again analogous to the case of fields. Given a Galois extension K/F and an intermediate field E, so $K \supseteq E \supseteq F$, let H be the corresponding subgroup of $\mathrm{Gal}(K/F)$. Then E/F is Galois iff $H \trianglelefteq \mathrm{Gal}(K/F)$.

$$\begin{array}{ccc} K & & 1 \\ | & & | \\ E & & H \\ | & & | \\ F & & \mathrm{Gal}(K/F) \end{array}$$

Example 6. Let $Y = X = \mathbb{C}^\times$, and consider the covering map

$$\begin{aligned} p : Y &\to X \\ z &\mapsto z^3 . \end{aligned}$$

Then $\mathrm{Deck}(Y/X) = \{\mathrm{id}_Y, \sigma, \sigma^2\}$, where

$$\begin{aligned} \sigma : Y &\to Y \\ z &\mapsto \zeta z , \end{aligned}$$

where ζ is a primitive third root of unity. Given $x_0 \in X$, then

$$p^{-1}(x_0) = \{y_0, y_1, y_2\} = \{\sqrt[3]{x_0}, \zeta\sqrt[3]{x_0}, \zeta^2\sqrt[3]{x_0}\}$$

where $y_j = \zeta^j\sqrt[3]{x_0}$. Thus we see that $\mathrm{Deck}(Y/X)$ acts transitively on $p^{-1}(x_0)$: for instance, $\sigma(y_0) = y_1$, and $\sigma^2(y_0) = y_2$. Thus $p : Y \to X$ is Galois.

We can extend the notion of Galois to nonconstant morphisms of Riemann surfaces. Let $F : Y \to X$ be a nonconstant morphism of Riemann surfaces, and let $R \subseteq X$ be its ramification values. Let $X^* = X \setminus R$ and $Y^* = Y \setminus F^{-1}(R)$. As we have seen, then $F|_{Y^*} : Y^* \to X^*$ is a covering map.

Definition 7. A nonconstant morphism $F : Y \to X$ of Riemann surfaces is Galois or normal if the restricted covering map $F|_{Y^*} : Y^* \to X^*$ is Galois.

Theorem 8. *Let X be a Riemann surface and suppose that*

$$P(T) = T^n + c_1 T^{n-1} + \cdots + c_n \in \mathcal{M}(X)[T]$$

is an irreducible polynomial of degree n. Then there exist a Riemann surface Y, a morphsim $F : Y \to X$ of degree n and a mermorphic function $f \in \mathcal{M}(Y)$ such that $(F^*P)(f) = 0$.

Definition 9. Such a triple (Y, F, f) is called the algebraic function defined by $P(T)$.

The proof of this result is the machinery of multi-valued functions. Here's a proof by example for the case $X = \mathbb{A}^1$ with coordinate S. Then the coefficients $c_1, \ldots, c_n$ are simply rational functions in S. Multiplying $P(T)$ by the least common denominator of $c_1, \ldots, c_n$, we obtain a polynomial $Q(S,T) \in \mathbb{C}[S,T]$. Let C be the curve in $\mathbb{A}^2$ given by $Q(S,T) = 0$. One can show that Q is irreducible, however C may have singular points. One can resolve these singular points and obtain a smooth curve $\widetilde{C}$, which is the desired Riemann surface.

Theorem 10. *Let X be a Riemann surface and $K := \mathcal{M}(X)$ be its field of meromorphic functions. Suppose $P(T) \in K[T]$ is an monic, irreducible polynomial of degree d.*

(a) Let (Y, π, F) be the algebraic function defined by $P(T)$ and let $L := \mathcal{M}(Y)$. Identifying K with its image under $\pi^* : K \to L$, then L/K is field extension of degree d and $L \cong K[T]/(P(T))$.

(b) Every deck transformation $\sigma \in \mathrm{Deck}(Y/X)$ induces an automorphism

$$\begin{aligned} \widehat{\sigma} : L &\to L \\ f &\mapsto \sigma \cdot f := f \circ \sigma^{-1} \end{aligned}$$

that fixes K, and the map

$$\begin{aligned} \mathrm{Deck}(Y/X) &\to \mathrm{Aut}(L/K) \\ \sigma &\mapsto \widehat{\sigma} \end{aligned}$$

is a group homomorphism, and in fact, an isomorphism.

(c) The covering $Y \to X$ is Galois if and only if the extension L/K of function fields is Galois.

Proof.

(a) The first statement follows from results above.

(b) To show that the map $\mathrm{Deck}(Y/X) \to \mathrm{Aut}(L/K)$ is an isomorphism requires more results about deck transformations, but we can at least show it's a homomorphism. Given $\sigma, \tau \in \mathrm{Deck}(Y/X)$ and $f \in \mathcal{M}(Y)$, then

$$\begin{aligned} \widehat{\sigma \circ \tau}(f) &= (\sigma \circ \tau) \cdot f = f \circ (\sigma \circ \tau)^{-1} = f \circ \tau^{-1} \circ \sigma^{-1} = \widehat{\sigma}(f \circ \tau^{-1}) \\ &= \widehat{\sigma}(\widehat{\tau}(f)) = \widehat{\sigma} \circ \widehat{\tau}(f) \end{aligned}$$

so $\widehat{\sigma \circ \tau} = \widehat{\sigma} \circ \widehat{\tau}$.

(c) (Assume we know part (b) is true.) Fix a basepoint $x_0 \in X$, and let $\pi^{-1}(x_0) = \{y_1, \dots, y_d\}$. Then $\pi : Y \to X$ is Galois iff for each $j = 1, \dots, d$ there exists a deck transoformation $\sigma_j \in \mathrm{Deck}(Y/X)$ such that $\sigma_j(y_1) = y_j$. (This is glossing over some details, but they follow from uniqueness of lifts for covering maps.) Thus $\pi : Y \to X$ is Galois iff $\# \mathrm{Deck}(Y/X) = d$. Similarly, L/K is Galois iff $\# \mathrm{Aut}(L/K) = [L : K] = d$. Since $\mathrm{Deck}(Y/X) \cong \mathrm{Aut}(L/K)$ by the previous part, then the result follows.

□

Example 11. Let $E : y^2 = x^3 - x$ be an elliptic curve and $\pi : E \to \mathbb{P}^1$ be the projection $(x,y) \mapsto x$. Then the corresponding extension of function fields is

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \mathcal{M}(E) = \dfrac{\mathbb{C}(x)[y]}{(y^2 - (x^3 - x))} \arrow[d, dash, "\pi^*"] \\ \mathbb{P}^1 & \mathbb{C}(x)\,. \end{tikzcd}$$

The deck transformations of π are the identity and the hyperelliptic involution $\iota : (x, y) \mapsto (x, -y)$, and the corresponding field automorphism is

$$\begin{aligned} \iota^* : \mathcal{M}(E) &\to \mathcal{M}(E) \\ f(x,y) &\mapsto f(x,-y)\,. \end{aligned}$$

Thus we see that the covering is Galois: given $x_0 \in \mathbb{P}^1$, the points in the fiber π^{-1} are simply (x_0, y_0) and $(x_0, -y_0)$, where y_0 is a solution to $y^2 - (x_0^3 - x_0)$, and these points are exchange by the involution ι.

We also see directly that the extension of function fields is Galois, as $[\mathcal{M}(E) : \mathbb{C}(x)] = 2$ and $\mathrm{Aut}(\mathcal{M}(E)/\mathbb{C}(x)) = \{\mathrm{id}, \iota^*\}$, so $\# \mathrm{Aut}(\mathcal{M}(E)/\mathbb{C}(x)) = 2$.

Definition 12. A function field in one variable is a finite extension of $\mathbb{C}(x)$, the field of rational functions with coefficients in $\mathbb{C}$.

Proposition 13. *There is an equivalence of categories between the category of compact, connected Riemann surfaces, whose arrows are morphisms of Riemann surfaces, and the category of function fields in one variable, whose arrows are field monomorphisms.*

III. Uniformization of Riemann surfaces

III.1. Universal covers of Riemann surfaces.

Theorem 14. *Every simply connected Riemann surface is isomorphic to exactly one of the following: (i) $\mathbb{P}^1$, (ii) $\mathbb{C}$, or (iii) $\mathfrak{D}$.*

This means that if X is a Riemann surface, its universal cover $\widetilde{X}$ must be one of these three. Moreover, letting $G = \mathrm{Deck}(\widetilde{X}/X)$, then $X \cong G \backslash \widetilde{X}$, so every Riemann surface can be expressed as a quotient of either the Riemann sphere, the complex plane, or the unit disc.

Theorem 15 (Uniformization of compact, connected Riemann surfaces)**.** *According to their universal coverings, compact, connected Riemann surfaces can be classified as follows:*

- $\mathbb{P}^1$ *is the only compact Riemann surface of genus* 0.
- *Every compact, connected Riemann surface of genus* 1 *is isomorphic to* $\mathbb{C}/\Lambda$ *for some full lattice* $\Lambda \leq \mathbb{C}$.
- *Every compact, connected Riemann surface of genus* ≥ 2 *is isomorphic to a quotient* $\Gamma \backslash \mathfrak{H}$ *for some subgroup* $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ *acting freely and properly discontinuously on* $\mathfrak{H}$.

III.2. $\mathrm{PSL}_2(\mathbb{R})$ as the group of isometries of hyperbolic space.

Definition 16. The hyperbolic metrix on the upper half-plane $\mathfrak{H}$ with coordinate $z = x + iy$ is defined by

$$\frac{|dz|^2}{(\mathrm{Im}(z))^2} := \frac{(dx)^2 + (dy)^2}{y^2}\,.$$

The notation $|dz|^2$ is used for the Euclidean metric $(dx)^2 + (dy)^2$ because it hints at its transformation property: transforming $|dz|^2$ by a holomorphic map f results in $|f'(z)|^2 |dz|^2$.